SMALL OSCILLATIONS OF A PENDULUM HAVING A SPHERICAL CAVITY FILLED WITH A VISCOUS FLUID

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The oscillations about an immoveable axis of an axisymmetric body having a spherical cavity filled with viscous incompressible fluid are considered. It is assumed that the center of the cavity lies on the axis of symmetry of the body and that the axis of rotation intersects the axis of symmetry.

The frequency equation for small oscillations is derived.

1. We introduce an immoveable coordinate system XYZ, whose origin O is at the intersection of the axes of symmetry and rotation. The Z-axis is directed vertically downward and the Y-axis is along the direction of no motion. An xyz coordinate system is associated with the body with its origin O at the center of the spherical cavity. The z-axis is directed downwards along the axis of symmetry, and the x and y axes are placed in a perpendicular plane so that in a position of equilibrium they are parallel to the corresponding axes of the immoveable system (Fig.1). The position of the body is determined by the angle δ_1 between the Z and the z axes.

Let $\mathbf{V}$ be the absolute velocity of the fluid particles $\mathbf{U}$ their velocity relative to the body; then

$$\mathbf{V} = \mathbf{\Omega} \times \mathbf{r} + \mathbf{U}$$

where $\mathbf{\Omega}$ is the vector of angular velocity of the body and $\mathbf{r}$ is the radius vector of the fluid particle relative to an immoveable point. The linearized equation for kinetic moment of the system body and fluid about the Y-axis has the form

$$A\delta_1'' + Qa\delta_1 + \gamma \frac{d}{dt} \int_\tau (zU_x - xU_z)\, d\tau + \gamma b \frac{d}{dt} \int_\tau U_x \, d\tau = 0$$

$$(A = A_1 + A_2) \qquad (1)$$

Here A_1 is the moment of inertia of the shell about the Y-axis and A_2 is the moment of inertia of the fluid mass; Q is the weight of the system; a is the distance of the center of gravity of the system from the immoveable axis; γ is the fluid density; τ is the cavity volume; and b is the distance of the cavity center from the immoveable axis.

2. The motion of the fluid in the body cavity is referred to the xyz system of the body. The linearized equations of the relative motion as projections on the axes of the moveable system xyz have the form

$$\frac{\partial U_x}{\partial t} + \delta_1'' z = -\frac{\partial P_1}{\partial x} + \nu \Delta U_x$$
$$\frac{\partial U_y}{\partial t} = -\frac{\partial P_1}{\partial y} + \nu \Delta U_y \qquad (2)$$
$$\frac{\partial U_z}{\partial t} - \delta_1'' x = -\frac{\partial P_1}{\partial z} + \nu \Delta U_z$$

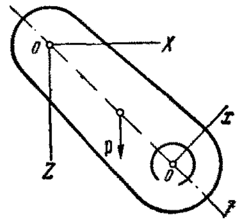

Fig. 1

Here

$$P_1 = \frac{1}{\gamma} P - \frac{1}{2} (\delta')^2 (x^2 + z^2 + 2bz) + b\delta'' x$$

If the equation of continuity and the boundary conditions

$$\operatorname{div} U = 0, \qquad U|_S = 0 \qquad (3)$$

are added to Equations (2), where S is the surface of the spherical cavity, then the entire system describes the motion of the fluid in the body cavity.

In Equations (1), (2) and (3) we pass from the xyz system to spherical coordinates ρ, θ, φ and introduce complex functions u_ρ, u_θ, u_φ, p and δ such that

$$U_\rho = \operatorname{Re}(u_\rho), \quad U_\theta = \operatorname{Re}(u_\theta), \quad U_\varphi = \operatorname{Re}(u_\varphi), \quad P_1 = \operatorname{Re}(p), \quad \delta_1 = \operatorname{Re}(\delta)$$

and combinations of the complex velocities are

$$u_+ = -{}^1/_2 \sqrt{2}\,(u_\varphi + iu_\theta), \qquad u_0 = u_\rho, \qquad u_- = {}^1/_2 \sqrt{2}\,(u_\varphi - iu_\theta)$$

3. We write the functions u_0, u_+, u_- and p in the form of series in generalized spherical functions [1] (4)

$$u_0 = \sum_{l=1}^{\infty} \sum_{n=-l}^{l} f_{0n}^{\,l}(\rho, t)\, T_{0n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0), \quad u_+ = \sum_{l=1}^{\infty} \sum_{n=-l}^{l} f_{1n}^{\,l}(\rho, t)\, T_{1n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0)$$

$$u_- = \sum_{l=1}^{\infty} \sum_{n=-l}^{l} f_{-1n}^{\,l}(\rho, t)\, T_{-1n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0), \quad p = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} F_n^{\,l}(\rho, t)\, T_{0n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0)$$

$$T_{mn}^{\,l}({}^1/_2\pi - \varphi, \theta, 0) = P_{mn}^{l}(\cos\theta)\, e^{-in({}^1/_2\pi - \varphi)} \qquad (m = -1, 0, 1)$$

Here $f_{mn}^{\,l}$, $F_n^{\,l}$ are unknown functions of ρ and t.

Upon substitution of the series (4) into the equations describing the fluid motion, it is easy to convince oneself that the motion of the body is affected only by motions described by terms in the series with indices $l = 1$, $n = \pm 1$. Consequently, for a study of the body motion it is sufficient to find the functions

$$f_{0-1}^{1},\ f_{01}^{1},\ f_{-1-1}^{1},\ f_{-11}^{1},\ f_{+1-1}^{1},\ f_{11}^{1},\ F_{-1}^{1},\ F_{1}^{1}$$

We seek them in the form of series in the natural functions of the problem associated with the oscillations of a viscous fluid in an immoveable vessel. These natural functions are solutions of Equations [2]

$$-k^2 \mathbf{w} = -\nu^{-1} \operatorname{grad} p + \Delta \mathbf{w}, \qquad \operatorname{div} \mathbf{w} = 0$$

with the boundary condition $\mathbf{w}|_S = 0$.

The zero solution of the equations has the form of (4), where

$$f_{0n}^{1} = \frac{C_{1n}}{\nu k^2} + C_{2n} \frac{J_{3/2}(k\rho)}{(k\rho)^{3/2}}, \qquad F_n^{1} = C_{1n}\rho, \qquad C_{1n},\ C_{2n},\ C_n = \text{const}$$
$$(5)$$
$$f_{\pm 1n}^{1} = -\frac{C_{1n}}{\nu k^2} + \frac{C_{2n}}{2}\left[\frac{J_{1/2}(k\rho)}{(k\rho)^{1/2}} - 2\frac{J_{3/2}(k\rho)}{(k\rho)^{3/2}}\right] \pm C_n \frac{J_{3/2}(k\rho)}{(k\rho)^{1/2}}$$

Upon satisfying the boundary condition $\mathbf{w}|_S = 0$ and setting C_{1n} and C_{2n} equal to zero, since the terms in (5) that contain these constants do not affect the body motion, we look for u_0, u_+, u_- in the form

$$u_0 = 0$$

$$u_\pm = \pm \sum_{j=1}^{\infty} \frac{J_{3/2}(k_j\rho)}{(k_j\rho)^{1/2}} \left[C_{-1}^{\,j}(t)\, T_{\pm 1-1}\left(\frac{\pi}{2} - \varphi, \theta, 0\right) + C_1^{\,j}(t)\, T_{\pm 11}\left(\frac{\pi}{2} - \varphi, \theta, 0\right)\right] \qquad (6)$$

where k_j is a positive root of the Bessel function $J_{3/2}(kR)$ or of Equation

$$\tan kR = kR \qquad (7)$$

4. For determination of the functions $C_{-1}^{\,j}(t)$ and $C_1^{\,j}(t)$ we substitute solution (6) into the equations of fluid motion and equate coefficients of the same functions $T_{mn}^{\,1}(^1/_2\pi - \varphi, \theta, 0)$. Then we obtain

$$\sum_{j=1}^{\infty} \varphi_j \left(\frac{dC_n^{\,j}}{dt} + \nu k_j^2 C_n^{\,j}\right) = \frac{1}{\sqrt{2}}\rho\delta'' \qquad \left(\varphi_j = \frac{J_{3/2}(k_j\rho)}{(k_j\rho)^{1/2}}\right)$$

where the k_j, are positive roots of Equation (7) and $n = \pm 1$.

By expansion of ρ in a series in φ_j, substitution of this series in the preceding equation, and equating coefficients of φ_j, we get

$$\frac{dC_n^{\,j}}{dt} + \nu k_j^2 C_n^{\,j} - \delta'' \sqrt{\pi}\, \frac{\sqrt{1 + (k_j R)^2}}{k_j} = 0 \qquad (j = 1, 2, \ldots, \infty;\ n = \pm 1) \qquad (8)$$

To these equations must be added the equation of body motion

$$A\delta'' + Qa\delta - \frac{8}{3}\gamma \sqrt{\pi} R^3 \sum_{j=1}^{\infty} \frac{1}{k_j \sqrt{1 + (k_j R)^2}} \left(\frac{dC_{-1}^{\,j}}{dt} + \frac{dC_1^{\,j}}{dt}\right) = 0 \qquad (9)$$

We seek a solution to the system (8) and (9) proportional to $e^{\lambda t}$. We then obtain a characteristic equation which after simple transformations takes the form

$$0.1\,\frac{Qa}{J}\,\frac{1}{\lambda^2} + 0.1\,\frac{A - J}{J} = -\sum_{j=1}^{\infty} \frac{1}{s_j^2 + R^2\nu^{-1}\lambda} \qquad (J = {}^8/_{15}\,\gamma\pi R^3) \qquad (10)$$

Here the notation J for the moment of inertia of the fluid mass about a cavity diameter has been introduced, as well as s_j for the positive roots of the equation $\tan s = s$, making use of the relation $s_1^{-2} + s_2^{-2} + s_3^{-2} + \ldots = 0.1$.

Equation (10) has a multiplicity of negative roots associated with a damped motion of the body, and a pair of complex conjugate roots associated with an oscillatory body motion.

If in Equation (10) we pass to the limit with $\nu \to \infty$ or $\nu \to 0$, we obtain the frequency equation either for oscillations of the body with a hardened fluid or for oscillations of the body filled with ideal fluid. It must be noted that Krasnoshchekov in a recent paper [3] considered an approximate method of solving the problem of the oecillating pendulum with a cavity containing viscous fluid.

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